Double jump in the maximum of two-type reducible branching Brownian motion

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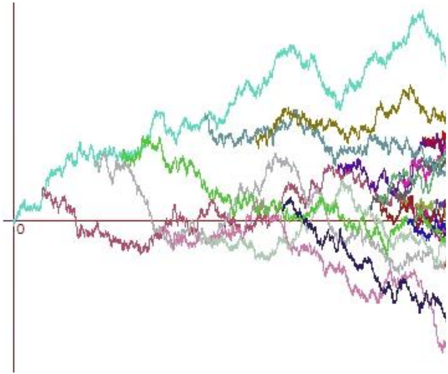
Based on the joint work with

Heng Ma (Peking University)

Standard Branching Brownian motions (BBMs)

- Initially a particle move as a standard Brownian motion.
- At rate 1 it splits into 2 particles.



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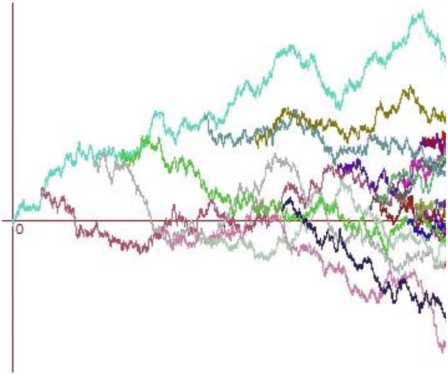
- These particles behave independently of each other, continue move and split, subject to the same rule.



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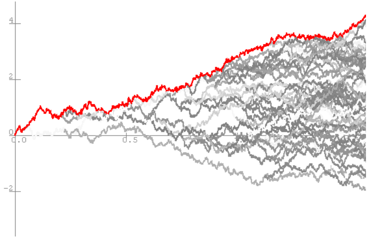
Denote the process by $(\mathsf{X}_i(t))_{i=1}^{n(t)}$. Let $\mathsf{M}_t := \max_{i \leq n(t)} \mathsf{X}_i(t)$ be the maximal displacement among all the particles alive at time t.

- **Biggins'76**: $\lim_{t\to\infty} \frac{M_t}{t} = \sqrt{2}$ a.s.



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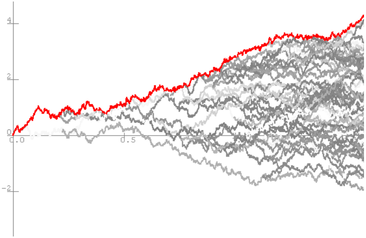

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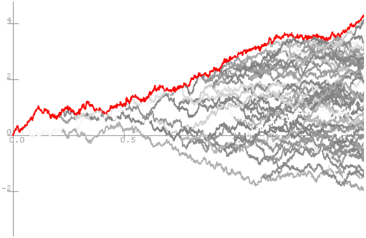
- **Lalley-Sellke'87**: The limiting distribution is a randomly shifted Gumbel distribution: There exist constant $C_\star$ and random variable Z_∞ such that

$$\lim_{t\to\infty} \mathsf{P}(\mathsf{M}_t - m_t \le x) = \mathsf{E}[\exp\{-C_\star \mathsf{Z}_\infty e^{-\sqrt{2}x}\}].$$



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- **Aïdékon-Berestycki-Brunet-Shi'13**, **Arguin-Bovier-Kistler'13**: The *extremal process* $\sum_{i \le n(t)} \delta_{\mathsf{x}_i(t)-m(t)}$ converges in distribution to a certein decorated Poisson point process (DPPP):

$$\sum_{i \le n(t)} \delta_{\mathsf{x}_i(t)-m(t)} \Rightarrow \mathrm{DPPP}(\sqrt{2} C_\star \mathsf{Z}_\infty e^{-\sqrt{2}x} dx, \mathfrak{D}^{\sqrt{2}}).$$

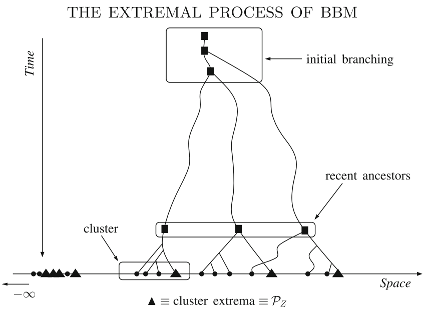

Figure 3: Construction of the limiting extremal process

Universality

BBM is perhaps the simplest model in the universality class called log-correlated Fields.

- BRW (**Addario-Berry & Reed'09**, **Hu-Shi'09**, **Aïdékon'13**, **Bramson-Ding-Zeitouni'16**, **Madaule'17**)

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- 2DGFF (**Bramson-Zeitouni'12**, **Bramson-Ding-Zeitouni'16**, **Biskup-Louidor'16**, **Biskup-Louidor'18**) For $m_N = \sqrt{2/\pi}(2\log N - \frac{3}{4}\log\log N)$,

$$\lim_{N\to\infty} P\left(\max_{v\in V_N} X_v^N \leq m_N + x\right) = E\left[e^{-CZe^{-\frac{2}{\sqrt{g}}x}}\right].$$

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- Cover times of 2D torus by Brownian motion (**Dembo-Peres-Rosen-Zeitouni'04**, **Belius-Kistler'17**)
- High-values of the Riemann zeta-function (**Arguin-Belius-Harper'17**, **Arguin-Belius-Bourgade-Radziwiłł-Soundararajan'19**, **Arguin-Dubach-Hartung'21+**)
- $\cdots\cdots$

Variants of BBM are also received many attention.

- Variable speed BBM.(**Fang-Zeitouni'12**, **Bovier-Hartung'14**, **Bovier-Hartung'15**, **Mallein'15**, **Maillard-Zeitouni'16**, **Bovier-Hartung'20**)

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- d-dimensional BBM (**Mallein'15**, **Stasiński-Berestycki-Mallein'22**, **Kim-Lubetzky-Zeitouni'23**, **Berestycki- Kim-Lubetzky-Mallein-Zeitouni'21+**.)

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- Multi-type (irreducible) BBM. (**Biggins'76**, **R.-Yang'14**, **Hou-R.-Song'23+**)
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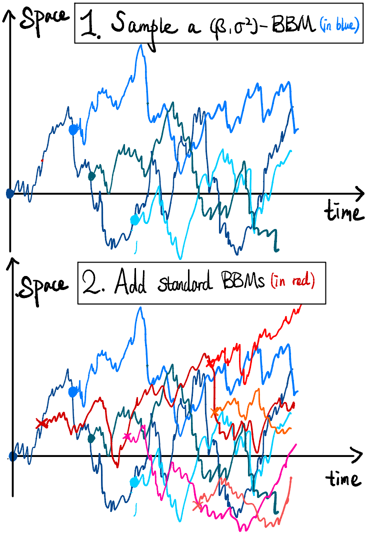
Multi-type branching Brownian motion

In a two-type reducible branching Brownian motion:



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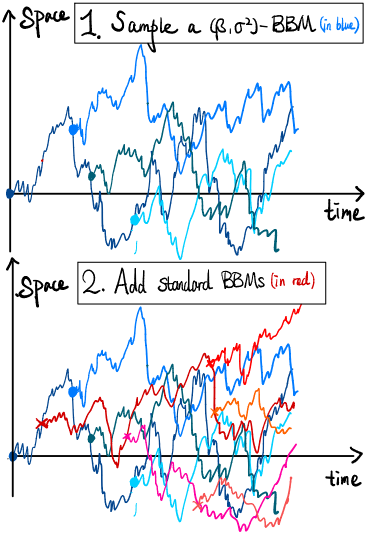
- Type 1 particles move as Brownian motion with diffusion coefficient σ^2. They split at rate β into two children of type 1; and give brith to type 2 particles at rate α.



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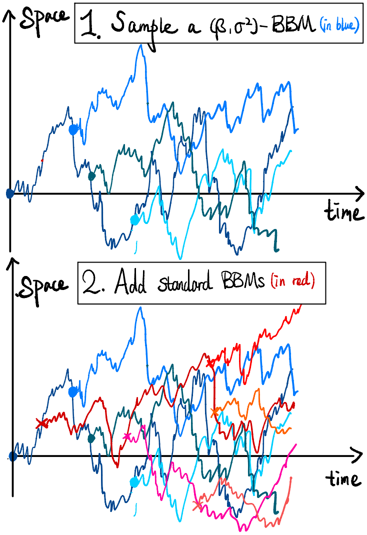
- Type 2 particles move as standard Brownian motion and branch at rate 1 into two type 2 children, but can not produce children of type 1.



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- Asymtotic behavior of the extremal process $\sum_{i=1}^{n(t)} \delta_{X_i(t)-m(t)}$

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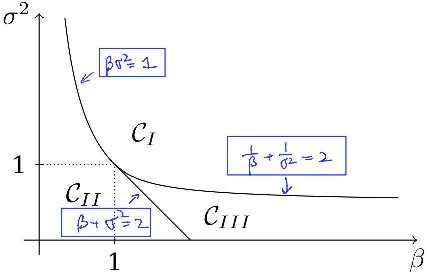
Biggins'12: Speed of the multi-type branching process in a more general setting.



Remark: Those asymptotic behaviour do not depend on α.

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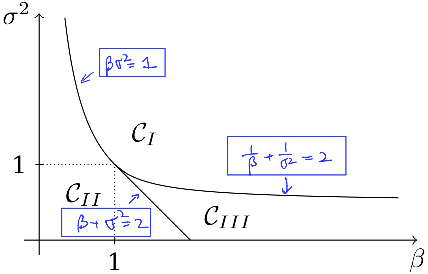

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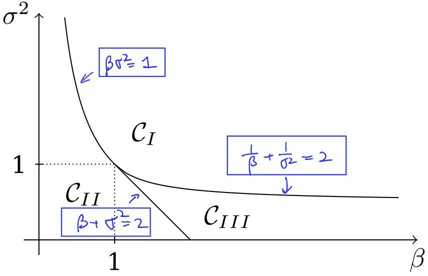

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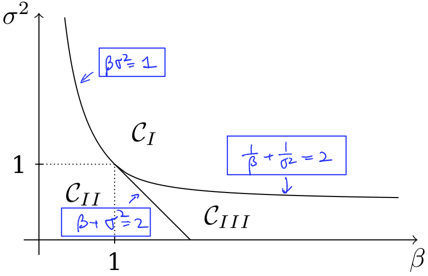
- When $(\beta, \sigma^2) \in \mathcal{C}_{III}$, anomalous spreading occurs: $\frac{M_t}{t} \to v^* = \frac{\beta-\sigma^2}{\sqrt{2(1-\sigma^2)(\beta-1)}}$. The speed of the two-type process is strictly larger than the speed of both single type particle systems.



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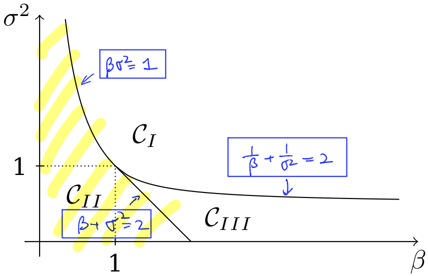
Asymptotic behaviour of extremal particles

- **Belloum-Mallein'21**: When $(\beta, \sigma^2) \in \mathcal{C}_{II}$,

$$M_t - (\sqrt{2}t - \frac{3}{2\sqrt{2}} \log t)$$

converges in law. Moreover,

$$\sum_{i \leq n(t)} \delta_{X_i(t) - \sqrt{2}t + \frac{3}{2\sqrt{2}} \log t} \Rightarrow \mathrm{DPPP}\left(\sqrt{2} C_\star \bar{Z}_\infty e^{-\sqrt{2}x} dx, \mathfrak{D}^{\sqrt{2}}\right)$$



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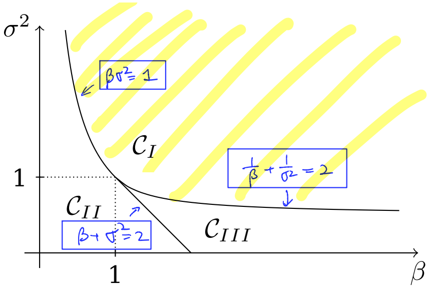


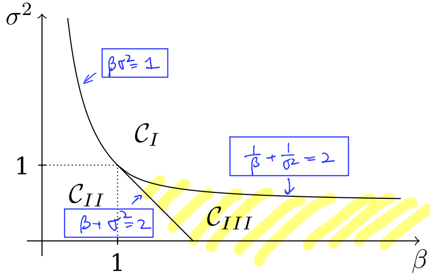
Asymptotic behaviour of extremal particles

- **Belloum-Mallein'21**: When $(\beta, \sigma^2) \in \mathcal{C}_{III}$,

$$M_t - v^* t$$

converges in law. Moreover, for $\theta^* = \sqrt{2\frac{\beta-1}{1-\sigma^2}}$

$$\sum_{i \leq n(t)} \delta_{X_i(t) - v^* t} \Rightarrow \mathrm{DPPP}\left(\theta^* C W_\infty^{\beta,\sigma^2}(\theta^*) e^{-\theta^* x} dx, \mathfrak{D}^{\theta^*}\right)$$



- **Belloum'22+**: When $(\beta, \sigma^2) = (1,1)$, then

$$M_t - (\sqrt{2}t - \frac{1}{2\sqrt{2}} \log t)$$

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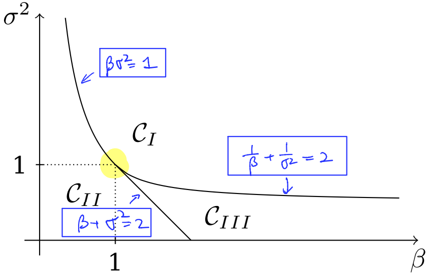
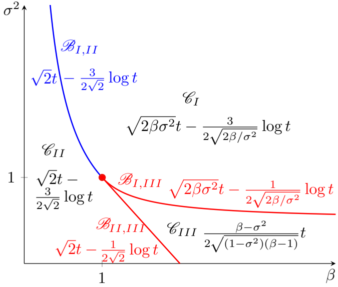


Our results

Denote by $\mathscr{B}_{I,II} = \partial\mathcal{C}_I \cap \partial\mathcal{C}_{II} \backslash \{(1,1)\}$. Define similarly $\mathcal{B}_{II,III}$.

Theorem (Ma-R.'23+)



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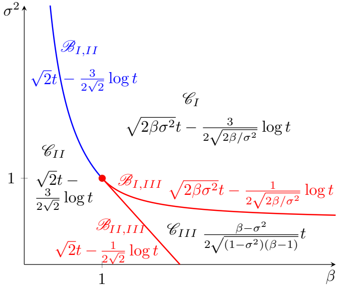

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- *When* $(\beta, \sigma^2) \in \mathscr{B}_{II,III}$*. Let*

$$m_t^{2,3} := \sqrt{2}t - \frac{1}{2\sqrt{2}} \log t.$$

Then $M_t - m_t^{2,3}$ *converges in law. Moreover,*

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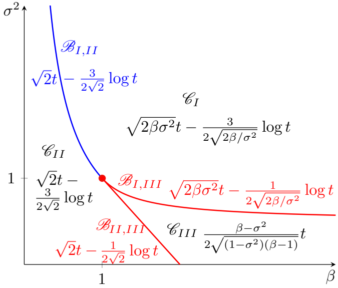

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$$m_t^{1,3} := \sqrt{2\beta\sigma^2}t - \frac{1}{2\theta}\log t$$

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$$\sum_{u\in N_t} \delta_{X_u(t)-m_t^{1,3}} \Rightarrow$$

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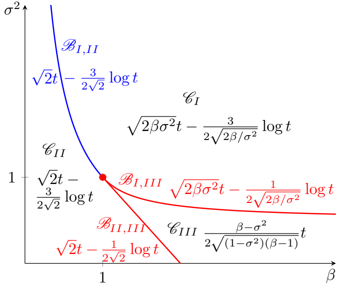

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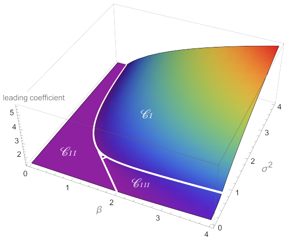

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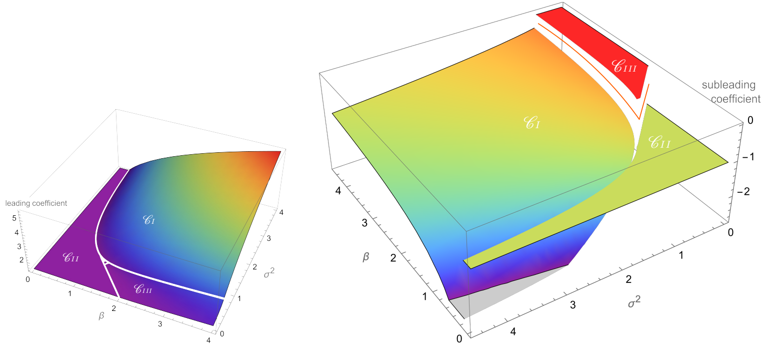

A double jump occurs in the maximum when (β, σ^2) cross the boundary of the anomalous spreading region. (See also **Fang-Zeitouni'12**, **Mallein'15**)

Some words about our proof

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Two key insights in our proof:

- localization of the extremal paths.
  For example, when $(\beta, \sigma^2) \in \mathscr{B}_{I,III}$, we showed that if u is an extremal particle at time t, then u is of type 2 and let T_u=the born time of its oldest type 2 ancestor, we should have

$$T_u = t - \Theta(\sqrt{t}) \quad \text{and}$$

$$X_u(T_u) = \sqrt{2\beta\sigma^2} T_u - (\theta - \sqrt{2\beta\sigma^2})(t - T_u) + \Theta(\sqrt{t - T_u}),$$

here

$$c_1\sqrt{t} \leq \Theta(\sqrt{t}) \leq c_2\sqrt{t}$$

Some words about our proof

- **CLT about the Gibbs measure (Madaule'16)**: For every bounded continuous function F,

$$\sum_{i\leq n(t)} F\left(\frac{\sqrt{2}t - X_i(t)}{\sqrt{t}}\right) \frac{e^{\sqrt{2}X_i(t)-2t}}{\sum_{i\leq n(t)} e^{\sqrt{2}X_i(t)-2t}} \to \langle F, \mu\rangle$$

in probability, where $\mu = ze^{-z^2/2} I_{(z>0)}$. In particular, taking $F = 1_{[a+\lambda, b+\lambda]}$, we have

$$\sum_{i\leq n(t)} 1_{\{\sqrt{2}t - X_i(t) \in [\lambda\sqrt{t} + a\sqrt{t}, \lambda\sqrt{t} + b\sqrt{t}]\}} \frac{e^{\sqrt{2}X_i(t)-2t}}{\sum_{i\leq n(t)} e^{\sqrt{2}X_i(t)-2t}}$$

$$\to \langle F, \mu\rangle. \quad \text{(CLT)}$$

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Gibbs measure:

$$\frac{1}{\sum_{i\leq n(t)} e^{-\sqrt{2}(\sqrt{2}t-X_i(t))}}\sum_{i\leq n(t)} e^{-\sqrt{2}(\sqrt{2}t-X_i(t))}\delta_{\sqrt{2}t-\mathsf{X}_i(t)}$$

Some words about our proof

- Local CLT about the Gibbs measure (Ma-R.'23+):
  Let G be a non-negative bounded measurable function with compact support. Suppose $F_t(z) = G(\frac{z-r_t}{h_t})$.

$$\sum_{i\leq n(t)} F_t\left(\frac{\sqrt{2}t - X_i(t)}{\sqrt{t}}\right)\frac{e^{\sqrt{2}X_i(t)-2t}}{\sum_{i\leq n(t)} e^{\sqrt{2}X_i(t)-2t}} \sim \langle F_t, \mu\rangle$$

in probability. In particular, we have

$$(i) \sum_{i\leq n(t)} 1_{\{\sqrt{2}t-X_i(t)\in[\lambda\sqrt{t}+a,\lambda\sqrt{t}+b]\}}\frac{e^{\sqrt{2}X_i(t)-2t}}{\sum_{i\leq n(t)} e^{\sqrt{2}X_i(t)-2t}}$$

$$\sim \frac{b-a}{\sqrt{t}}\lambda e^{-\frac{\lambda^2}{2}};$$

$$(ii) \sum_{u\in \mathsf{N}_t} 1_{\{\sqrt{2}t-X_i(t)\in[\lambda\sqrt{t}+at^{1/4},\lambda\sqrt{t}+bt^{1/4}]\}}\frac{e^{\sqrt{2}X_i(t)-2t}}{\sum_{i\leq n(t)} e^{\sqrt{2}X_i(t)-2t}}$$

$$\sim \frac{(b-a)}{t^{1/4}}\lambda e^{-\frac{\lambda^2}{2}}.$$

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END

Thank you!

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